On the convergence of the iterates of “FISTA”.

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Abstract

We discuss here the convergence of the iterates of the “FISTA” algorithm, which is an accelerated algorithm proposed by Beck and Teboulle for minimizing the sum $F = f + g$ of two convex, l.s.c. and proper functions, such that one is differentiable with Lipschitz gradient, and the proximity operator of the second is easy to compute. It builds a sequence of iterates $(x_n)_{n\in\mathbb{N}}$ for which $F(x_n) - F(x^*) \leqslant O(1/n^2)$. However the convergence of these iterates $(x_n)_{n\in\mathbb{N}}$ is not obvious. We show here that with a small modification, we can ensure the same decay of the energy as well as the (weak) convergence of the iterates to a minimizer.

Introduction

Let $\mathcal{H}$ be a Hilbert space and f and g two convex, l.s.c functions from $\mathcal{H}$ to $\mathbb{R}\cup\{+\infty\}$ such that f is differentiable with L-Lipschitz continuous gradient, and g is “simple”, meaning that its “proximal map”

$$x \mapsto \arg\min_{y\in\mathcal{H}} g(y) + \frac{\|x-y\|^2}{2\tau}$$

can be easily computed. We consider the following minimization problem

$$\min_{x\in\mathcal{H}} F(x) := f(x) + g(x) \tag{1}$$

and we assume that this problem has at least a solution (and possibly an infinite set of solutions).

Among the many algorithms which exist to tackle such problems, the proximal splitting algorithms, which perform alternating descents in f and in g, are frequently used, because of their simplicity and relatively small per-iteration complexity. One can mention the Forward-Backward (FB) splitting, the Douglas-Rachford splitting, the ADMM (alternating direction method of multipliers),[1] which all have been proved to be efficient in many imaging problem such as denoising, inpainting, deconvolution, color transfert and many others.

This work focuses on the so-called “Fast Iterative Soft Thresholding Algorithm” (FISTA) which is an accelerated variant of the Forward-Backward algorithm proposed by Beck and Teboulle [2], built upon ideas of Nesterov [12] and Güler [7].

The FB is a descent algorithm which defines a sequence $(x_n)_{n\in\mathbb{N}}$ by performing an explicit descent in f and implicit in g. It is then shown that there exist $C>0$, such that for all $n\in\mathbb{N}$ $F(x_n)-F(x^*)\leqslant\frac{C}{n}$ where x^* is a minimizer of F. Moreover the sequence $(x_n)_{n\in\mathbb{N}}$ weakly converges in $\mathcal{H}$. See for instance [14] or [2] for a simple derivation of this rate.

The sequence $(x_n)_{n\in\mathbb{N}}$ defined by the accelerated variant “FISTA” [2] satisfies, on the other hand, $F(x_n)-F(x^*)\leqslant\frac{C'}{n^2}$ for a suitable real number C', however no convergence of $(x_n)_{n\in\mathbb{N}}$ has been proved so far.

The FISTA algorithm is based on a simple over-relaxation step with varying parameter, and several choices of parameters yield roughly the same rate of convergence. This paper provides complementary results on the convergence of $F(x_n)-F(x^*)$ for some “good” choices of these parameters, for which the weak convergence of the iterates can also be proved.

In the next section, we introduce a few definitions and our main notation. In a second part, the main result on the convergence of FISTA is recalled, and we give new results on the convergence of the values of $F(x_n)$, for other over-relaxation sequences. In the third part we show the convergence of the iterates. This part is strongly inspired from a recent paper of Pock and Lorenz [9], inspired by works of Alvarez and Attouch [1] and Moufadi and Oliny [10]. The last part is focused on numerical experiments.

1 Notation and definitions

In the following x^* denotes a solution of (1), even if this solution is not unique the value $F(x^*)$ is uniquely defined.

A key tool of FISTA is the proximal map. To any proper, convex and l.s.c function h is associated the proximal map Prox_h which is a function from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$\text{Prox}_h(x)=\arg\min_{y\in\mathcal{H}} h(y)+\frac{1}{2}\|x-y\|^2 .$$

This function is uniquely defined and generalizes the projection on a closed convex set to convex functions.

[1] See for instance [4, 8, 5, 6, 3].

In the sequel, γ denotes a non negative real number such that $\gamma \leq \frac{1}{L}$ where L is the Lipschitz constant of ∇f and T the mapping from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$T(x) := \text{Prox}_{\gamma g}(x - \gamma \nabla f(x)),$$

The idea of FB is to apply this mapping from any $x_0 \in \mathcal{H}$ using Krasnosel'ski Mann iterations to get a weak convergence to a minimizer x^* of F. The idea of FISTA is to apply this mapping using a suitable extragradient rule to accelerate the convergence.

FISTA is defined by a sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 and a point $x_0 \in \mathcal{H}$. Let $(t_n)_{n\in\mathbb{N}^*}$ be a sequence of non negative real numbers and $x_0 \in \mathcal{H}$, the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are defined by $y_0 = u_0 = x_0$ and for all $n \geqslant 1$,

$$x_n = T(y_{n-1}) \tag{2}$$

$$y_n = \left(1 - \frac{1}{t_{n+1}}\right) x_n + \frac{1}{t_{n+1}} u_n \tag{3}$$

$$u_n = x_{n-1} + t_n(x_n - x_{n-1}). \tag{4}$$

The point y_n may also be defined from points x_n and x_{n-1} by

$$y_n = x_n + \alpha_n(x_n - x_{n-1}) \text{ with } \alpha_n := \frac{t_n - 1}{t_{n+1}} \tag{5}$$

For suitable choices of $(t_n)_{n\in\mathbb{N}^*}$ the sequence $(F(x_n))_{n\in\mathbb{N}}$ converge to $F(x^*)$, i.e the sequence $(w_n)_{n\in\mathbb{N}}$, defined as follows,

$$w_n := F(x_n) - F(x^*) \tag{6}$$

tends to 0 when n goes to infinity.

Several proofs use bounds on the local variation of the sequence $(x_n)_{n\in\mathbb{N}}$, which we will denote $(\delta_n)_{n\in\mathbb{N}}$: variation :

$$\delta_n := \frac{1}{2} \|x_n - x_{n-1}\|_2^2 \tag{7}$$

The sequence $(v_n)_{n\in\mathbb{N}}$ denoting the distance between u_n and a fixed minimizer x^* of F will also be useful:

$$v_n := \frac{1}{2} \|u_n - x^*\|_2^2 . \tag{8}$$

To complete this part devoted to our notation, we define a sequence $(\rho_n)_{n\in\mathbb{N}}$, associated to $(t_n)_{n\in\mathbb{N}^*}$, whose positivity will ensure the convergence of the FISTA iterations:

$$\rho_n := t_{n-1}^2 - t_n^2 + t_n. \tag{9}$$

2 Some results on the FISTA method

The main result of [2] is the following Theorem :

Theorem 1 ([2, Thm. 4.1])**.** *For any $x_0 \in \mathcal{H}$, if the sequence $(t_n)_{n\in\mathbb{N}^*}$ satisfies*

$$\forall n \geqslant 2 \quad t_n^2 - t_n \leqslant t_{n-1}^2 \tag{10}$$

and $t_1 = 1$, if $\gamma \leqslant \frac{1}{L}$ then the sequence $(x_n)_{n\in\mathbb{N}}$ satisfies for all $n \in \mathbb{N}$

$$w_n \leqslant \frac{1}{2\gamma t_n^2} \|x_0 - x^*\|_2^2 \tag{11}$$

for any minimizer x^* of F.

Condition (10) can also be stated using the sequence $(\rho_n)_{n\in\mathbb{N}}$: $\forall n \geqslant 2$, $\rho_n \geqslant 0$.

The sequence defined by $t_1 = 1$ and

$$\forall n \in \mathbb{N}^* \quad t_{n+1} = \sqrt{t_n^2 + \frac{1}{4}} + \frac{1}{2} \tag{12}$$

achieves the equality in (10).Also, it turns out that the sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_n = \frac{n+1}{2}$ satisfies condition (10). But more generally, for any $a \geqslant 2$ the sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_n = \frac{n+a-1}{a}$ satisfies (10). Indeed,

$$\rho_n = \frac{1}{a^2}((n+a-2)^2-(n+a-1)^2+a(n+a-1)) = \frac{1}{a^2}((a-2)n+a^2-3a+3) \geqslant 0. \tag{13}$$

An induction proves that any sequence satisfying (10) (hence an inequality in (12)) and $t_1 = 1$ satisfies $t_n \geqslant n$. Hence for any sequence defined above, Theorem 1 ensures that

$$\forall n \in \mathbb{N} \quad w_n \leqslant \frac{C}{n^2} \tag{14}$$

where C depends on the exact choice of the sequence $(t_n)_{n\in\mathbb{N}}$.

A priori, the best constant "C" in this bound $\frac{C}{n^2}$ will be reached if the sequence $(t_n)_{n\in\mathbb{N}}$ is the one achieving the equality in (10), given by (12), ensuring the highest value of t_n. This is the choice in [2], and it turns out that it is nearly optimal (since for any n there exists a problem which has lower bound of the same order, see [11, 13]). We will soon see, however, that not achieving this equality may have some advantages.

This first Theorem can easily be made more precise, as follows:

Theorem 2. *If the sequence $(t_n)_{n\in N}$ satisfies (10) and $t_1 = 1$, if $\gamma \leqslant \frac{1}{L}$ then for any $N \geqslant 2$,*

$$t_{N+1}^2 w_{N+1} + \sum_{n=1}^{N} \rho_{n+1} w_n \leqslant \frac{v_0 - v_{N+1}}{\gamma}. \tag{15}$$

Using the several choices of sequence $(t_n)_{n\in\mathbb{N}}$ described above, Theorem 2 ensures the same decay for w_n ($w_n \leqslant \frac{C}{n^2}$) as the previous. But using (13), we readily see that for a "good" choice of the sequence $(t_n)_{n\in\mathbb{N}^*}$, one obtains the following corollary:

Corollary 1. *Let $a > 2$ and for $n \geqslant 1$, $t_n = \frac{n+a-1}{a}$. Then the sequence $(nw_n)_{\in\mathbb{N}}$ belongs to $\ell_1(\mathbb{N})$. In particular, $\liminf_{n\to\infty} n^2 \log n w_n = 0$.*

The classical choice (12) for the sequence $(t_n)_{n\in\mathbb{N}^*}$ may yield the best rate of convergence for the objective, but other sequences can give better global properties of the sequence $(nw_n)_{n\in\mathbb{N}}$. (also notice that the other classical choice corresponding to $a = 2$ will not ensure this summability.) An important remark, here, is that this result is not in contradiction with the lower bounds of Nemirovski and Yudin (see [11, 13]). Indeed, they show that for any integer n_0 one can build a specific problem for which one will have, after n_0 iterations, $n_0 w_{n_0} \geq C/n_0$, however this does not mean that the sequence $(nw_n)_n$ is not eventually summable.

Proof of Theorem 2 The proof is similar to the one of Theorem 1 in [2], however for the ease of the reader we will sketch it here. A first (standard) technical descent Lemma is useful:

Lemma 1. *Let $\gamma \in]0, \frac{1}{L}]$, where L is the Lipschitz constant of ∇f, $\bar{x} \in \mathcal{H}$ and $\hat{x} = T\bar{x}$. Then*

$$\forall x \in \mathcal{H} \quad F(\hat{x}) + \frac{\|\hat{x} - x\|^2}{2\gamma} \leqslant F(x) + \frac{\|x - \bar{x}\|^2}{2\gamma} \tag{16}$$

Proof. Many proofs exist of this result, we give an elementary one which is inspired from [16]. By definition of the proximal map, $\hat{x}$ is the minimizer of the $\frac{1}{\gamma}$-strongly convex function

$$z \longmapsto g(z) + f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle + \frac{1}{2\gamma} \|z - \bar{x}\|^2$$

hence for all $z \in \mathcal{H}$

$$g(\hat{x}) + f(\bar{x}) + \langle \nabla f(\bar{x}), \hat{x} - \bar{x} \rangle + \frac{\|\bar{x} - \hat{x}\|^2}{2\gamma} + \frac{\|z - \hat{x}\|^2}{2\gamma} \leqslant g(z) + f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle + \frac{\|z - \bar{x}\|^2}{2\gamma}.$$

Since $\gamma \leq 1/L$, it follows

$$g(\hat{x}) + f(\hat{x}) + \frac{1}{2\gamma} \|z - \hat{x}\|^2 \leqslant g(z) + f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle + \frac{1}{2\gamma} \|z - \bar{x}\|^2 .$$

By convexity, $f(\bar{x}) + \langle \nabla f(\bar{x}), z - \bar{x} \rangle \leqslant f(z)$. We deduce that (16) holds and the Lemma is proved. $\square$

Proof of Theorem 2. Applying this Lemma to $\bar{x} = y_n$, $\hat{x} = x_{n+1}$ and $x = (1 - \frac{1}{t_{n+1}})x_n + \frac{1}{t_{n+1}}x^*$, we find

$$F(x_{n+1}) + \frac{\left\|\frac{1}{t_{n+1}}u_{n+1} - \frac{1}{t_{n+1}}x^*\right\|_2^2}{2\gamma} \leqslant F\left((1 - \frac{1}{t_{n+1}})x_n + \frac{1}{t_{n+1}}x^*\right) + \frac{\left\|\frac{1}{t_{n+1}}x^* - \frac{1}{t_{n+1}}u_n\right\|_2^2}{2\gamma}$$

Using the convexity of F it follows

$$F(x_{n+1}) - F(x^*) - \left(1 - \frac{1}{t_{n+1}}\right)(F(x_n) - F(x^*)) \leqslant \frac{\|u_n - x^*\|_2^2}{2\gamma t_{n+1}^2} - \frac{\|u_{n+1} - x^*\|_2^2}{2\gamma t_{n+1}^2}$$

Using definitions of w_n and v_n this inequality can be stated

$$t_{n+1}^2 w_{n+1} - (t_{n+1}^2 - t_{n+1})w_n \leqslant \frac{v_n - v_{n+1}}{\gamma} \tag{17}$$

Summing these inequalities from $n = 1$ to $n = N$ leads to

$$t_{N+1}^2 w_{N+1} + \sum_{n=1}^{N} \rho_{n+1} w_n \leqslant \frac{v_0 - v_{N+1}}{\gamma}. \tag{18}$$

which ends the proof of Theorem 2. □

We can deduce another useful corollary:

Corollary 2. *Let $a > 2$ and for $n \geqslant 1$, $t_n = \frac{n+a-1}{a}$. Then the sequence $(n\delta_n)_{\in\mathbb{N}}$ belongs to $\ell_1(\mathbb{N})$, in particular $\liminf_{n\to\infty} n^2 \log n\delta_n = 0$. In addition, there exists $C > 0$ such that for all $n \in \mathbb{N}^*$, $\delta_n \leqslant \frac{C}{n^2}$.*

This results which is a consequence of Corollary 1 is the key to prove the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$.

Proof. Applying Lemma 1 to $\bar{x} = y_n = x_n + \alpha_n(x_n - x_{n-1})$, and $x = x_n$ leads to

$$F(x_{n+1}) + \frac{\|x_n - x_{n+1}\|^2}{2\gamma} \leqslant F(x_n) + \frac{\alpha_n^2 \|x_n - x_{n-1}\|^2}{2\gamma}$$

which can be written with definitions of w_n and δ_n

$$\delta_{n+1} - \alpha_n^2 \delta_n \leqslant \gamma(w_n - w_{n+1})$$

If $t_n = \frac{n+a-1}{a}$, $\alpha_n = \frac{t_n - 1}{t_{n+1}} = \frac{n-1}{n+a}$.

Multiplying this inequality by $(n+a)^2$ and summing from $n = 1$ to $n = N$ leads to

$$\sum_{n=1}^{N}(n+a)^2(\delta_{n+1} - \alpha_n^2\delta_n) \leqslant \gamma\sum_{n=1}^{N}(n+a)^2(w_n - w_{n+1}),$$

which gives

$$(N+a)^2\delta_{N+1} + \sum_{n=2}^{N}((n+a-1)^2 - (n+a)^2\alpha_n^2)\delta_n \leqslant \\ \gamma\left((a+1)^2 w_1 - (N+a)^2 w_{N+1} + \sum_{n=2}^{N}\left((n+a)^2 - (n+a-1)^2\right)w_n\right)$$

that is

$$(N+a)^2\delta_{N+1} + \sum_{n=2}^{N}a(2n-2+a)\delta_n \\ \leqslant \gamma\left((a+1)^2 w_1 - (N+a)^2 w_{N+1} + \sum_{n=2}^{N}(2n+2a-1)w_n\right)$$

By Corollary 1 and since we have assumed $a > 2$, the right part of the inequality is uniformly bounded independently of N, which ensures that the sequence $(n\delta_n)_{n\in\mathbb{N}}$ belongs to $\ell_1(\mathbb{N})$. It also follows that $N^2\delta_{N+1}$ is globally bounded. □

3 Convergence of the iterates of FISTA

In this section, we show the following Theorem

Theorem 3. *Let $a > 2$ be a positive real number, and for all $n \in \mathbb{N}$ let $t_n = \frac{n+a-1}{a}$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ given by FISTA weakly converges to a minimizer of F.*

The proof of the theorem follows the ideas of Pock and Lorenz, in the proof of Theorem 1 in [9]—see also [1]. The two main differences between our setting and the setting of [9] are:

1. We do not assume the existence of $\alpha < 1$ such that $\forall n \geqslant 1$, $\alpha_n \leqslant \alpha$;
2. The sequence $(\delta_n)_{n\in\mathbb{N}}$ produced by FISTA, with a good choice of the sequence (t_n), has stronger properties than in [9].

It turns out that Corollary 2 is crucial, while classical bounds on δ_n which only show the existence of a constant $C > 0$ such that $\delta_n \leqslant \frac{C}{n^2}$ are not sufficient.

Before giving the complete proof of this result, several remarks can be done.

1. From Corollary 2 it follows that the sequence $(n(x_{n+1} - x_n))_{n\in\mathbb{N}}$ is bounded, moreover from (18) it follows that the sequence $(v_n)_{n\in\mathbb{N}}$ defined in (8) is also bounded (hence $(u_n)_{n\in\mathbb{N}}$). These two facts imply that the sequence $((x_n)_{n\in\mathbb{N}})$ is bounded, hence weakly sequentially compact.
2. Assume we have a subsequence which weakly converges to a $\tilde{x} \in \mathcal{H}$, $x_\nu \rightharpoonup \tilde{x}$: then since the sequence $(\delta_n)_{n\in\mathbb{N}}$ tends to 0, $y_\nu \rightharpoonup \tilde{x}$ which shows that $\tilde{x}$ is a fixed point of the nonexpansive operator T. Hence it is a minimizer of F.

If we are able to prove that the sequence $\|x_n - x^*\|$ has a limit for any minimizer x^* of F, Theorem 3 will follow, from points 1. and 2. above and the observation that if $x_\nu \rightharpoonup \tilde{x}$ and $x_{\nu'} \rightharpoonup \tilde{x}'$, then using $\lim_\nu \|x_\nu - \tilde{x}\|^2 = \lim_{\nu'} \|x_{\nu'} - \tilde{x}\|^2$ and the same equality with $\tilde{x}'$, it follows $\|\tilde{x} - \tilde{x}'\|^2 = 0$ (this is Opial's Theorem [15]). Before proving Theorem 3, let us establish the following estimate.

Lemma 2. *For all $j \geqslant 1$, let us define*

$$\beta_{j,k} = \prod_{l=j}^{k} \alpha_l = \prod_{l=j}^{k} \frac{l-1}{l+a},$$

for all $k \geqslant j$, and $\beta_{j,k} = 1$ for $k < j$. (Observe that since $\alpha_1 = 0$, $\forall k > 1, \beta_{1,k} = 0$.) Then, we have for all j

$$\sum_{k=j}^{+\infty} \beta_{j,k} \leqslant \frac{j+5}{2}. \tag{19}$$

Proof. Since $a \geq 2$,

$$\beta_{j,k} \leqslant \prod_{l=j}^{k} \frac{l-1}{l+2}.$$

Hence, for all $j \geqslant 2$ and for all $k \geqslant 1$, $\beta_{j,k} \leqslant 1$, while if $k - j \geqslant 2$,

$$\beta_{j,k} \leqslant \left(\frac{j+1}{k}\right)^3.$$

It follows that for all $j \geqslant 2$,

$$\sum_{k=j}^{+\infty} \beta_{j,k} \leqslant 2 + \sum_{k=j+2}^{+\infty} \beta_{j,k} \leqslant 2 + \sum_{k=j+2}^{+\infty} \left(\frac{j+1}{k}\right)^3 \leqslant 2 + (j+1)^3 \sum_{k=j+2}^{+\infty} \frac{1}{k^3}$$
$$\leqslant 2 + (j+1)^3 \int_{t=j+1}^{+\infty} \frac{dt}{t^3} \leqslant 2 + (j+1)^3 \frac{1}{2(j+1)^2}.$$

Estimate (19) follows. □

Proof of Theorem 3. Let us define

$$\Phi_n = \frac{1}{2}\|x_n - x^*\|_2^2 \quad \text{and} \quad \Gamma_n = \frac{1}{2}\|x_{n+1} - y_n\|_2^2$$

From the identity

$$\langle a-b, a-c\rangle = \frac{1}{2}\|a-b\|_2^2 + \frac{1}{2}\|a-c\|_2^2 - \frac{1}{2}\|b-c\|_2^2 \tag{20}$$

we have by using the definition of y_n

$$\Phi_n - \Phi_{n+1} = \delta_{n+1} + \langle y_n - x_{n+1}, x_{n+1} - x^*\rangle - \alpha_n \langle x_n - x_{n-1}, x_{n+1} - x^*\rangle \tag{21}$$

Then, using the monoticity of ∂g, we deduce that for any $z_{n+1} \in \partial g(x_{n+1})$ and for any $z^* \in \partial g(x^*)$

$$\langle \gamma z_{n+1} - \gamma z^*, x_{n+1} - x^*\rangle \geqslant 0$$

By definition of x^*, $-\nabla(f(x^*)) \in \partial g(x^*)$ and $y_n - x_{n+1} - \gamma\nabla f(y_n) \in \gamma\partial g(x_{n+1})$.

It follows

$$\langle y_n - x_{n+1} - \gamma\nabla f(y_n) + \gamma\nabla f(x^*), x_{n+1} - x^*\rangle \geqslant 0$$
$$\langle y_n - x_{n+1}, x_{n+1} - x^*\rangle + \gamma\langle \nabla f(x^*) - \nabla f(y_n), x_{n+1} - x^*\rangle \geqslant 0$$

Combining with (21) we obtain

$$\Phi_n - \Phi_{n+1} \geqslant \delta_{n+1} + \gamma\langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - x^*\rangle - \alpha_n\langle x_n - x_{n-1}, x_{n+1} - x^*\rangle. \tag{22}$$

From the co-coercivity of ∇f, we have

$$\begin{aligned}
\langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - x^*\rangle
&= \langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - y_n + y_n - x^*\rangle \\
&\geqslant \frac{1}{L}\|\nabla f(y_n) - \nabla f(x^*)\|_2^2 + \langle \nabla f(y_n) - \nabla f(x^*), x_{n+1} - y_n\rangle \\
&\geqslant \frac{1}{L}\|\nabla f(y_n) - \nabla f(x^*)\|_2^2 - \frac{1}{L}\|\nabla f(y_n) - \nabla f(x^*)\|_2^2 - \frac{L}{2}\Gamma_n \\
&\geqslant -\frac{L}{2}\Gamma_n.
\end{aligned}$$

Substituting back into (22), we get

$$\Phi_n - \Phi_{n+1} \geqslant \delta_{n+1} - \frac{\gamma L}{2}\Gamma_n - \alpha_n \langle x_n - x_{n-1}, x_{n+1} - x^* \rangle,$$

and invoking (20) it follows that

$$\begin{aligned}\Phi_{n+1} - \Phi_n - \alpha_n(\Phi_n - \Phi_{n-1}) \leqslant &- \delta_{n+1} + \frac{\gamma L}{2}\Gamma_n \\ &+ \alpha_n(\delta_n + \langle x_n - x_{n-1}, x_{n+1} - x_n \rangle) \\ &= -\Gamma_n + \frac{\gamma L}{2}\Gamma_n + (\alpha_n + \alpha_n^2)\delta_n,\end{aligned}$$

where we have used the fact that

$$\delta_{n+1} - \alpha_n \langle x_n - x_{n-1}, x_{n+1} - x_n \rangle = \alpha_n^2 \frac{\|x_n - x_{n-1}\|^2}{2} - \frac{\|x_{n+1} - y_n\|^2}{2}.$$

Using $\frac{\alpha_n + \alpha_n^2}{2} \leqslant \alpha_n$ we obtain

$$\Phi_{n+1} - \Phi_n - \alpha_n(\Phi_n - \Phi_{n-1}) \leqslant -\left(1 - \frac{\gamma L}{2}\right)\Gamma_n + 2\alpha_n \delta_n \tag{23}$$

with $1 - \frac{\gamma L}{2} > 0$.

Now defining $\theta_n = \max(0, \Phi_n - \Phi_{n-1})$ we obtain

$$\theta_{n+1} \leqslant \alpha_n(\theta_n + 2\delta_n) \tag{24}$$

Applying recursively (24) it follows that for all $n \geqslant 2$ ($\alpha_1 = 0$, and in particular $\theta_1, \theta_2 = 0$).

$$\theta_{n+1} \leqslant 2 \sum_{j=2}^{n} \left(\prod_{l=j}^{n} \alpha_l \right) \delta_j = 2 \sum_{j=2}^{n} \beta_{j,n} \delta_j. \tag{25}$$

Hence (using (19)),

$$\begin{aligned}\sum_{n=2}^{+\infty} \theta_n &\leqslant 2 \sum_{n=1}^{+\infty} \sum_{j=2}^{n} \beta_{j,n} \delta_j \\ &\leqslant 2 \sum_{j=2}^{\infty} \delta_j \sum_{n=j}^{\infty} \beta_{j,n} \\ &\leqslant 2 \sum_{j=1}^{\infty} \delta_j \frac{j+5}{2}.\end{aligned}$$

From Corollary 2 the right side of the last inequality is finite if $a > 2$, therefore the sequence $(\theta_n)_{n \in \mathbb{N}}$ belongs to $\ell_1(\mathbb{N})$.

The end of the proof follows Lorenz and Pock [9]. We set $s_n = \Phi_n - \sum_{i=1}^n \theta_i$ and since $\Phi_n \geqslant 0$ and $\sum_{i=1}^n \theta_i$ is bounded independently of n, we see that s_n is bounded from below. 0n the other hand

$$s_{n+1} = \Phi_{n+1} - \theta_n - \sum_{i=1}^{n} \theta_n \leqslant \Phi_{n+1} - \Phi_{n+1} + \Phi_n - \sum_{i=1}^{n} \theta_i = s_n$$

and hence $(s_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence and thus is convergent. This implies that Φ_n is convergent, which concludes the proof of Theorem 3. □

4 Numerical Experiments

In the previous parts, it was shown that non classical choices of the sequence $(t_n)_{n\in\mathbb{N}}$ ensure weak convergence of iterates $(x_n)_{n\in\mathbb{N}}$ and good properties for the sequence $(F(x_n)-F(x^*))_{n\in N}$. On three examples, inpainting, deblurring and denoising, we compare several choices of parameters.

For each example the 4 following sequences are tested :

- $t_1 = 1$ and $\forall n \in \mathbb{N}$, $t_{n+1} = \sqrt{t_n^2 + \frac{1}{4}} + \frac{1}{2}$,
- $t_n = \frac{n+a+1}{a}, \forall n \in \mathbb{N}$ with $a = 2$, 3 and 4.

For each problem, at each iteration n, the values $\|x_n - x_{n-1}\|_2^2$ and $F(x^n) - F(x^*)$ are computed. Since $F(x^*)$ can not be exactly computed, the value $F(x^*)$ is estimated by the the minimum of the values computed on 2000 iterations for the four methods. The plot of these two quantities is thus given from $n = 1$ to $n = 1800$.

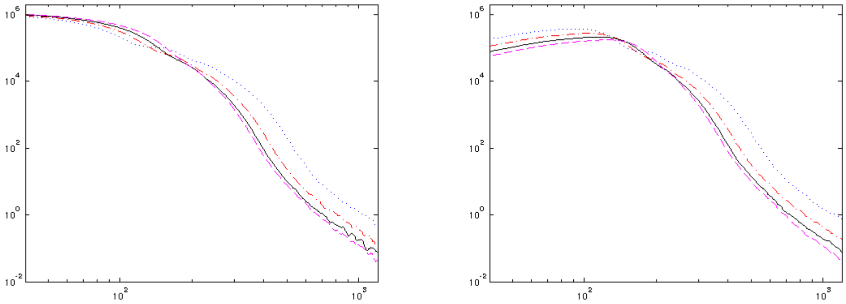
Inpainting Let us consider here a degraded image $y^0 = Mx^0$ where x^0 is an unkown source image and M a mask operator. In our example 50% of the pixels are removed. We estimate the image x^0 from y^0 by minimizing

$$F(x) = \frac{1}{2} \left\|y^0 - Mx\right\|_2^2 + \lambda \left\|Tx\right\|_1 \tag{26}$$

where λ is a small positive parameter and T an orthogonal (Daubechies) wavelet transform.

Considering $f(x) = \frac{1}{2} \left\|y^0 - Mx\right\|_2^2$ and $g(x) = \lambda \left\|Tx\right\|_1$, FISTA may be applied to minimize F.

Left: the masked image y^0. Right: an image $\hat{x}$ estimated minimizing F with FISTA.[2]



Left: values of $F(x_n) - F(x^*)$. Right: values of $\|x_n - x_{n-1}\|_2^2$. Blue dot line classical FISTA, red dashed-dot line $a = 2$, black solid line $a = 3$ and magenta dashed line $a = 4$.

On this example, the choices $a = 3$ or $a = 4$ seems better than classical choices after 100 iterations. One can notice that classical FISTA is better for a small number of iterations.

Deblurring In this second example $y^0 = h \star x^0 + n$ is the noisy image of a blurred images x^0, where h is a gaussian filter and n is a random gaussian noise. The image x^0 can be estimated minimizing

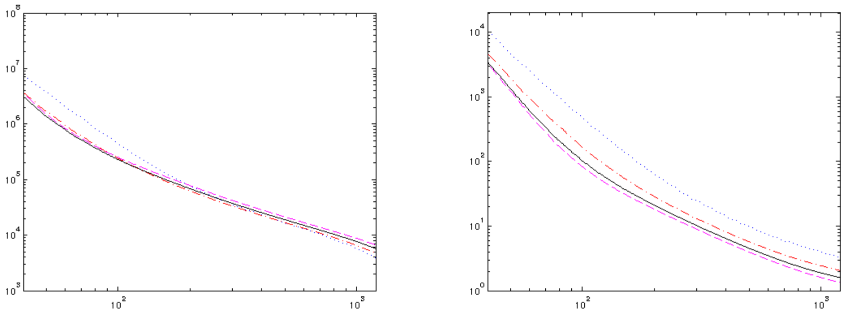
$$F(x) = \frac{1}{2} \left\|y^0 - h \star x\right\|_2^2 + \lambda \left\|Tx\right\|_1 \tag{27}$$

where λ is a small positive real number whose value depends on the noise level and T is an orthogonal (Daubechies) wavelet transform.

Considering $f(x) = \frac{1}{2} \left\|y^0 - h \star x\right\|_2^2$ and $g(x) = \lambda \left\|Tx\right\|_1$, FISTA may be applied to minimize F.

[2]The images provided by the 4 versions of FISTA look very similar, the difference appears in the values of the variation of functionnal F through the iterations.

Left: the blurred image y^0. Right: an image $\hat{x}$ estimated minimizing F with FISTA.



Left: values of $F(x_n) - F(x^*)$. Right: values of $\|x_n - x_{n-1}\|_2^2$. Blue dot line classical FISTA, red dashed-dot line $a = 2$, black solid line $a = 3$ and magenta dashed line $a = 4$.

On this example, the choices of the classical FISTA seems better after 200 iterations, but the decreasing of δ_n seems still better for $a = 3$ and $a = 4$.

TV denoising Let us consider now a noisy image $y^0 = x^0 + n$. The image x^0 may be estimated from y^0 minimizing

$$F(x) = \frac{1}{2}\left\|y^0 - x\right\|_2^2 + \lambda \left\|\nabla x\right\|_1 \tag{28}$$

where ∇x is the gradient of the image x and $\|\nabla x\|_1$ is the isotropic ℓ_1-norm of the gradient. This regularization is also called Total Variation (TV) regularization. The proximal map of the function $x \mapsto \|\nabla x\|_1$ does not have a close form and FISTA is difficult to use directly here. Nevertheless, by duality, this minimization problem is equivalent to minimize

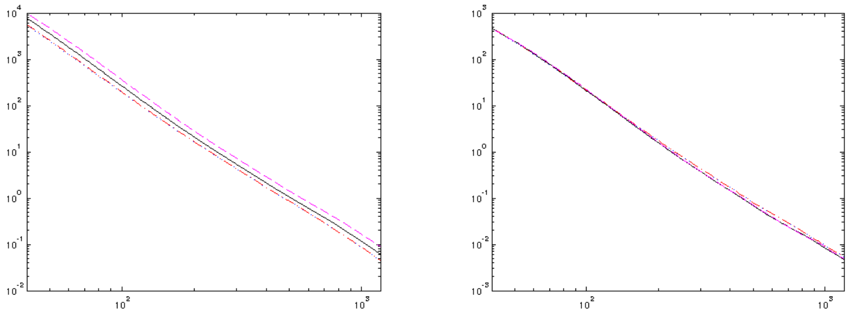
$$G(p) = \frac{1}{2}\left\|y^0 + \text{div } p\right\|_2^2 + i_{\|\cdot\|_\infty \leqslant \lambda}(p) \tag{29}$$

where i_C denotes the function such that $i_C(x) = 0$ if $x \in C$ and $i_C(x) = +\infty$ if $x \notin C$ and where $x \mapsto -\text{ div } x$ is the divergence operator, conjugate of

the gradient ∇.
These two problems are equivalent and for any solution p^* of the second minimization problem, $y^0 + \text{div } p^*$ is a solution of the first minimization problem.

This second problem can be solved using FISTA since the gradient of $p \mapsto \frac{1}{2} \|y + \text{div p}\|_2^2$ is Lipschitz and the proximal map of $p \mapsto i_{\|\cdot\|_{+\infty} \leqslant \lambda}(p)$ is a simple projection.

Left: noisy image y^0. Right: the image $\hat{x}$ estimated minimizing F with FISTA.



Left: values of $G(p_n) - G(p^*)$. Right: values of $\|p_n - p_{n-1}\|_2^2$.
Blue dot line classical FISTA, red dashed-dot line $a = 2$, black solid line $a = 3$ and magenta dashed line $a = 4$.

On this example, the different choices of parameters seem to be equivalent.

These three examples shows that choosing a priori a sequence $(t_n)_{n\in\mathbb{N}}$ for FISTA is difficult and that for a given problem, it would be useful to test various options. Sometimes the classical parameters proposed by Beck and Teboulle are better to get a faster minimization, sometimes the use of $a = 3$ or $a = 4$ is better. But on the three examples the norm of the variation δ_n is smaller for $a = 3$ or $a = 4$ than $a = 2$ or the classical FISTA, which may indicate that the convergence of the iterates is faster.

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